Dimensional Analysis

YORK COLLEGE & UNIVERSITY CENTRE

Why do we use dimensional analysis?

- It allows us to convert between different units for international cooperation

- We can simplify equations

- We can understand the relationship between different variables in a system

SI Units/Fundamental Variables

- **Length [L]** – Measures distance or size (example: meters)
- **Mass [M]** – Measures how much matter something has (example: kilograms)
- **Time [T]** – Measures the passage of events (example: seconds)
- **Electric Current [I]** – Measures the flow of electric charge (example: amperes)
- **Temperature [Θ]** – Measures heat or thermal energy (example: Celsius)
- **Amount of Matter [N]** – Measures quantity of particles (example: Moles)
- **Luminous Intensity [J]** – Measures the brightness of a light (example: Candelas)

Dimensions

- All variables can be understood in terms of fundamental variables
- We can break down even the most complex units into fundamental values

For example:

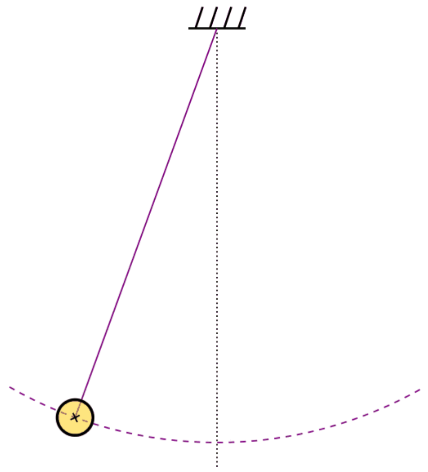
Velocity = Distance ÷ Time

Distance = Length [L]
Time = Time [T]

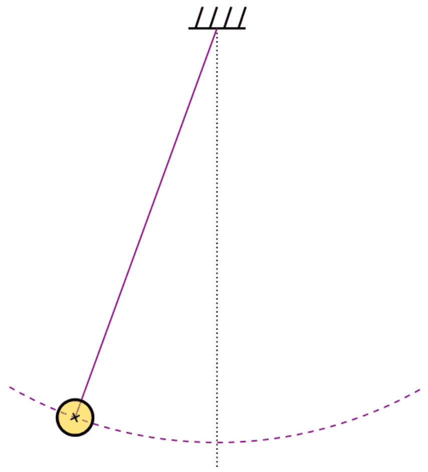
Units of velocity = [L/T]

Example of breaking down a complex variable

- We know that the variables effecting the period of a swinging pendulum are:
- $T \propto L * g$
- Where:
- **L** = Length of pendulum
- **g** = Acceleration due to gravity
- **T** = Time period



Example of breaking down a complex variable

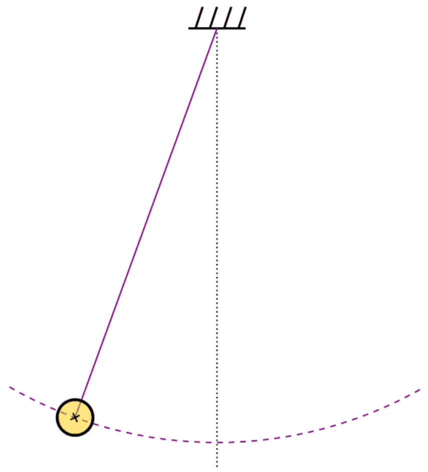


- **L** = Length of pendulum
- **g** = Acceleration due to gravity
- **T** = Time period

- Length is already in its fundamental form [L]
- Time is already in its fundamental form [T]

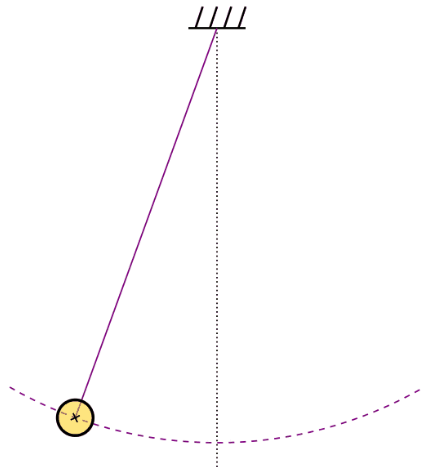
- Acceleration is not in its fundamental form, so we must zoom in on it

Example of breaking down a complex variable



- Acceleration is not in its fundamental form, so we must zoom in on it
- Acceleration = Distance/Time^2
- Distance is a **length [L]** value
- Time is already in its fundamental form [T]
- So, acceleration = $[L/T^2]$

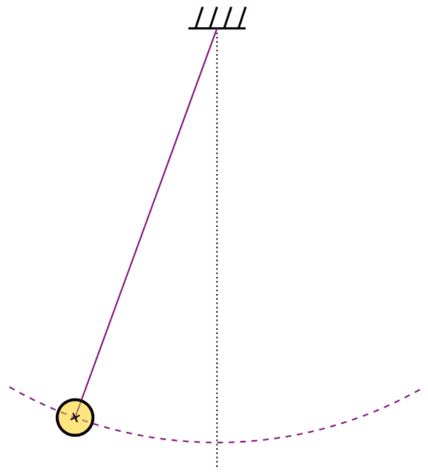
Example of breaking down a complex variable



- So, acceleration = $[L/T^2]$
- T ∝ L * g
- [T] ∝ [L] * $[L/T^2]$
- We can put it in an easier to read form by using the fact $\frac{a}{x^b} = ax^{-b}$
- [T] ∝ [L] $* [L * T^{-2}]$

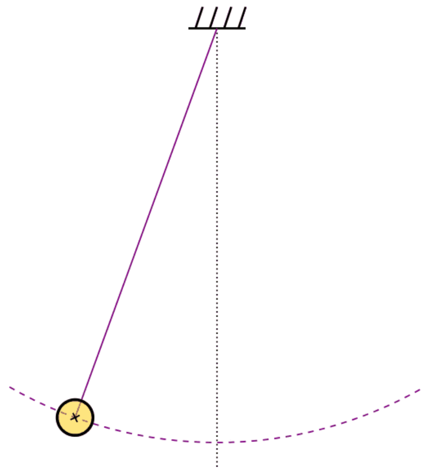
Example of finding an equation

- [T] ∝ [L] $* [L * T^{-2}]$ <- This is not the equation for the pendulum, it still needs work
- We need to make sure that the exponents are the same on each side as currently it is not true
- We need to split it again into its components and assign it exponents
- $[T^1] \propto [L^a] * [L^b * T^{-2b}]$
- We can neaten up by combining our L terms
- $[T^1] \propto [L^{a+b}] * [T^{-2b}]$

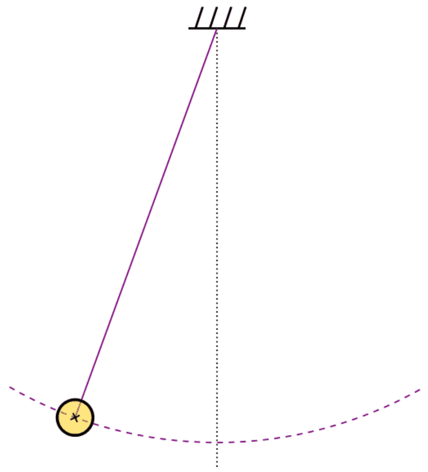


Example of finding an equation

- $[T^1] \propto [L^{a+b}] * [T^{-2b}]$
- We need to look at our terms on either side
- We only have T on both sides, and L only on one side
- We can set them equal
- $T^1 = T^{-2b}$
- $1 = L^{a+b}$



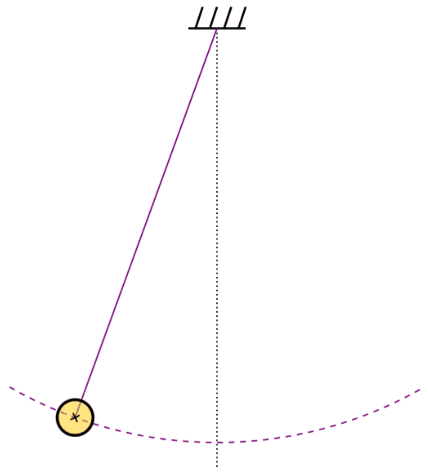
Example of finding an equation



- $T^{1} = T^{-2b}$
- $L^{0} = L^{a+b}$
- For T we can work out:
- $1 = -2b$
- Therefore, we know $b = -\frac{1}{2}$

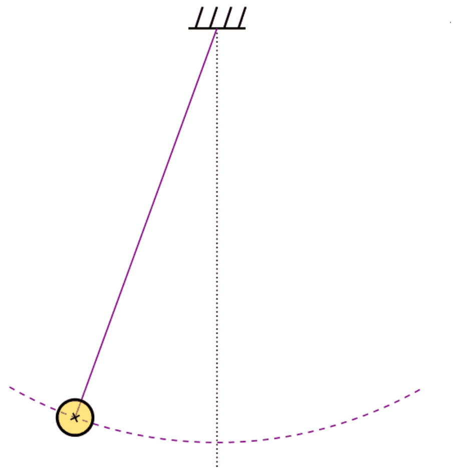
Example of finding an equation

- $b = -\frac{1}{2}$
- $1 = L^{a+b}$
- So $a - \frac{1}{2} = 0$ because $x^0 = 1$
- $a = \frac{1}{2}$
- $T \propto L^{1/2} * g^{-1/2}$



Example of finding an equation

- $T \propto L^{1/2} * g^{-1/2}$
- We know $x^{1/2} = \sqrt{x}$
- $\mathrm{a}x^{-b} = \frac{a}{x^b}$
- $T \propto \sqrt{\frac{L}{g}}$
- With experimentation we could then work out any missing constants, for this we are missing 2π



$$T = 2\pi\sqrt{\frac{L}{g}}$$

Dimensionless groups

In certain scenarios groups of variables can be put together in a group that cancels out, one example is the Reynolds number:

$$[Re] = \frac{[M/L^3] \cdot [L/T] \cdot [L]}{[M/LT]} = [1]$$

=

$$[Re] = \frac{\left[M \cdot L^{-3}\right] \cdot \left[L \cdot T^{-1}\right] \cdot [L]}{\left[M \cdot L^{-1} \cdot T^{-1}\right]}$$

$$Re = \frac{\rho v D}{\mu}$$

where:

- p = fluid density [M/L^3]
- v = velocity [L/T]
- D = pipe diameter [L]
- µ = viscosity [M/LT]

$$[Re] = M \cdot L^{-3} \cdot L \cdot T^{-1} \cdot L \cdot M^{-1} \cdot L^{1} \cdot T^{1}$$

As we can see they all cancel out to be $[\text{Value}]^0$ which equals 1 so its 1 * 1 * 1 which equals 1

What is a dimensionless group/number

- Dimensionless groups are groups of fundamental variables that cancel each other out
- This means that there are the same amount of positive and negative exponentials for each variable
- This means they are always equal to 1 no matter what you do

Why Dimensionless Groups Are Important

- **Simplify complex problems**→ Collapse multiple variables into fewer dimensionless parameters
- **Check similarity and scaling**→ Allows small-scale experiments (e.g., wind tunnels, water tanks) to represent full-scale systems
- **Universal results**→ Dimensionless numbers (e.g., Reynolds, Mach, Froude) apply across different systems and units
- **Highlight dominant effects**→ Show whether inertia, viscosity, gravity, or other forces control the system behaviour
- **Enable comparison**→ Engineers worldwide can compare results without worrying about unit systems

Example of a dimensionless group

- **The Reynolds number is one of the most widely used dimensionless groups, particularly in the study of fluid dynamics.**
- It is represented by [Re]
- It has the variables:
  - p = fluid density [M/L^3]
  - v = velocity [L/T]
  - D = pipe diameter [L]
  - µ = viscosity [M/LT]

$$Re = \frac{\rho v D}{\mu}$$

Proving Reynolds Number

- We can put our dimensions into the equation to get our dimensional format

$$Re = \frac{\rho v D}{\mu} = \frac{[M/L^3] \cdot [L/T] \cdot [L]}{[M/LT]}$$

- We then use the negative exponent rule to bring terms onto either side of the divide

$$= \frac{\left[M \cdot L^{-3}\right] \cdot \left[L \cdot T^{-1}\right] \cdot [L]}{\left[M \cdot L^{-1} \cdot T^{-1}\right]}$$

- We then do that again to get all the terms onto one line

$$= M \cdot L^{-3} \cdot L \cdot T^{-1} \cdot L \cdot M^{-1} \cdot L^{1} \cdot T^{1}$$

Proving Reynolds Number

- Finally, we collect like-terms, and we should see all variables are to the power of 0
- Therefore as 1*1*1 = 1 it must be dimensionless

$$= M \cdot L^{-3} \cdot L \cdot T^{-1} \cdot L \cdot M^{-1} \cdot L^{1} \cdot T^{1}$$

$$M^1 * M^{-1} = M^{1-1} = M^0 = 1$$

$$L^{-3} * L^1 * L^1 * L^1 = L^{-3+1+1+1} = L^0 = 1$$

$$T^{-1} * T^1 = T^{-1+1} = T^0 = 1$$

Buckingham π Theorem

- We wont always have easily found dimensionless groups
- So sometimes we have to find our own groups
- We can find out how many dimensionless groups an equation has using the Buckingham π Theorem

$$k = n - r$$

n = total number of variables
r = the number of fundamental variables

An example of working out dimensionless groups

Let's use the Buckingham π theorem to analyse the drag force (F_d) acting on a sphere moving through a fluid.

The drag force (F_d) depends on:

1. Fluid velocity (V)
2. Fluid density (ρ)
3. Fluid viscosity (μ)
4. Sphere diameter (D)

We aim to find the dimensionless groups (π terms) that describe the relationship.

Step 1: List the Variables and Their Dimensions

F_d : Drag force [M*L/T^2]
V : Velocity [L/T]
ρ : Density [M/L^3]
μ : Viscosity [M/L*T]
D : Diameter [L]

Step 2: Count Variables and Fundamental Dimensions and apply the theorem

Variables: 5 (F_d, V, ρ, μ, D)
Fundamentals: 3 (M, L, T)

k = n – r
K = 5 – 3
K = 2

An example of working out dimensionless groups (Continued)

Note learning repeating variables takes time, you learn what variables are common in a subject area

Step 3: Find repeating variables

Common in fluid dynamics are ρ, V and D

First dimensionless group often contains the dependent variable (the variable which we are studying, in this case it is F_d)

Group 1: F_d, ρ, V and D
Group 2: μ, ρ, V and D

Step 4: Work out the first dimensionless group

Group 1: $F_d * \rho * V * D$
Group 1: $[M*L/T^2] * [M/L^3]^a * [L/T]^b * [L]^c$
Group 1: $M * L * T^{-2} * M^a * L^{-3a} * L^b * T^{-b} * L^c$

For M: 1 + a = 0 -> a = -1
For L: 1-3a+b+c = 0
For T: -2-b = 0 -> b = -2

Put a and b into L to get c:
1+3-2+c = 0
c = -2

Group 1: $F_d * \rho^a * V^b * D^c$
Group 1: $F_d * \rho^{-1} * V^{-2} * D^{-2}$

$$\pi_1 = \frac{F_d}{\rho V^2 D^2}$$

An example of working out dimensionless groups (Continued)

Step 5: Work out the second dimensionless group

Group 2: μ * ρ * V * D
Group 2: $[M/L*T] * [M/L^3]^a * [L/T]^b * [L]^c$
Group 2: $M * L^{-1} * T^{-1} * M^a * L^{-3a} * L^b * T^{-b} * L^c$

For M: 1 + a = 0 -> a = -1
For L: -1-3a+b+c = 0
For T: -1-b = 0 -> b = -1

Put a and b into L to get c:
-1+3-1+c = 0
c = -1

Group 2: $\mu * \rho^a * V^b * D^c$
Group 2: $\mu * \rho^{-1} * V^{-1} * D^{-1}$

$$\pi_2 = \frac{\mu}{\rho V D}$$

Step 6: Check each group is dimensionless by writing it out

Step 7: Write out the relationship

$$\pi_1 = f(\pi_2)$$

$$\frac{F_d}{\rho V^2 D^2} = f\left(\frac{\mu}{\rho V D}\right)$$

Understanding dimensionless groups(Continued)

Step 8: Analyse the dimensionless groups we get (this step isn't necessary but is good to help understand)

$$\pi_1 = \frac{F_d}{\rho V^2 D^2}$$

Area

$$\pi_2 = \frac{\mu}{\rho V D}$$

This is the inverse Reynolds number (1/Re)